Math 22: Linear Algebra and Applications

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Placement Test
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1.1 Systems of Linear Equations

Definition 1.1.1 (Linear Equation)
A **linear equation** in the variables $x_1, x_2, \dots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \dots, a_n$ and b are constants, called the **coefficients** and **constant term**, respectively.

For example, $3x_1 - 2x_2 + 5x_3 = 7$ is a linear equation in the variables x_1, x_2, x_3 with coefficients $3, -2, 5$ and constant term 7.

Definition 1.1.2 (System of Linear Equations)
A **system of linear equations** is a collection of one or more linear equations involving the same set of variables.

Definition 1.1.3 (Solution)
A **solution** to a system of linear equations is an assignment of values to the variables that satisfies all equations in the system simultaneously.

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5.1 Eigenvectors and Eigenspaces

Definition 5.1.1 (Eigenvector and Eigenvalue)
Let A be an $n \times n$ matrix. A nonzero vector $\mathbf{v} \in \mathbb{R}^n$ is called an **eigenvector** of A if there exists a scalar λ such that

$$A\mathbf{v} = \lambda\mathbf{v}.$$

The scalar λ is called the **eigenvalue** corresponding to the eigenvector $\mathbf{v}$.

Theorem 5.1.2
The eigenvalues of a triangular matrix are the entries on its main diagonal.

Proof. For simplicity, let's consider the 3×3 matrix. If A is upper triangular, then $A - \lambda I$ has the form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ 0 & a_{22} - \lambda & a_{23} \\ 0 & 0 & a_{33} - \lambda \end{bmatrix}$$

The scalar λ is an eigenvalue of A if and only if $A - \lambda I$ is not invertible, which occurs if and only if the determinant is zero:

$$\det(A - \lambda I) = 0.$$

□

Lemma 5.1.3
Let A be a $n \times n$ matrix. Then A is invertible if and only if 0 is not an eigenvalue of A.

Theorem 5.1.4
If $v_1, v_2, \dots, v_k$ are eigenvectors of a matrix A corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$, then the set $\{v_1, v_2, \dots, v_k\}$ is linearly independent.

5.2 The Characteristic Equation

Example 5.2.1
Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$$

Solution. □

Definition 5.2.2 (Determinant)

The **determinant** of a square matrix A is a scalar value that can be computed from the elements of A and encodes certain properties of the linear transformation described by A. The determinant is denoted as $\det(A)$ or $|A|$.

Example 5.2.3

Compute $\det(A)$ for the matrix

$$\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Solution. □

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