Math 23: Differential Equations

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Placement Test
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This is my preparation for the placement test at Dartmouth College before Matriculation so that I can enroll in Math 54: Topology, and the prerequisite is Math 23: Differential Equations. I am following the syllabus from Winter 2024 to self-study.

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2 First Order Differential Equations

1 Linear Equations: Method of Integrating Factors

Definition 2.1 (First-order Linear Differential Equation)
A **first-order linear differential equation** is an equation that can be written in the standard form:

$$y' + p(t)y = g(t)$$

where $p(t)$ and $g(t)$ are given functions of the independent variable t. The equation is called **linear** because the dependent variable y and its derivative y' appear to the first power and are not multiplied together.

If $g(t) = 0$, the equation is called **homogeneous**:

$$y' + p(t)y = 0$$

Otherwise, it is called **non-homogeneous**.

Fact 2.2 (Method of Integrating Factors)
To solve the first-order linear differential equation $y' + p(t)y = g(t)$:

1. Calculate the **integrating factor**: $\mu(t) = e^{\int p(t)dt}$
2. Multiply both sides of the equation by $\mu(t)$:

$$\mu(t)y' + \mu(t)p(t)y = \mu(t)g(t)$$

3. Recognize that the left side is the derivative of $\mu(t)y$:

$$\frac{d}{dt}[\mu(t)y] = \mu(t)g(t)$$

4. Integrate both sides:

$$\mu(t)y = \int \mu(t)g(t)dt + C$$

5. Solve for y:

$$y = \frac{1}{\mu(t)}\left[\int \mu(t)g(t)dt + C\right]$$

Example 2.3 (Standard Form Example)
Solve the differential equation $y' + 2ty = t^2$.

Solution. This is already in standard form with $p(t) = 2t$ and $g(t) = t^2$.

Step 1: Find the integrating factor:

$$\mu(t) = e^{\int 2tdt} = e^{t^2}$$

Step 2: Multiply the equation by $\mu(t)$:

$$e^{t^2}y' + 2te^{t^2}y = t^2e^{t^2}$$

Step 3: Recognize the left side as a derivative:

$$\frac{d}{dt}[e^{t^2}y] = t^2e^{t^2}$$

Step 4: Integrate both sides:

$$e^{t^2}y = \int t^2e^{t^2}dt$$

Using integration by parts or substitution, we get:

$$e^{t^2}y = \frac{1}{2}e^{t^2}(t^2 - 1) + C$$

Step 5: Solve for y:

$$y = \frac{1}{2}(t^2 - 1) + Ce^{-t^2}$$

□

Example 2.4 (Equation Not in Standard Form)

Solve the differential equation $(4 + t^2)\frac{dy}{dt} + 2ty = 4t$.

Solution. First, we need to put this in standard form by dividing by $(4 + t^2)$:

$$\frac{dy}{dt} + \frac{2t}{4 + t^2}y = \frac{4t}{4 + t^2}$$

Now we have $p(t) = \frac{2t}{4+t^2}$ and $g(t) = \frac{4t}{4+t^2}$.

Step 1: Find the integrating factor:

$$\mu(t) = e^{\int \frac{2t}{4+t^2}dt}$$

Let $u = 4 + t^2$, then $du = 2tdt$:

$$\int \frac{2t}{4 + t^2}dt = \int \frac{1}{u}du = \ln|u| = \ln(4 + t^2)$$

Therefore: $\mu(t) = e^{\ln(4+t^2)} = 4 + t^2$

Step 2: Multiply by the integrating factor:

$$(4 + t^2)\frac{dy}{dt} + 2ty = 4t$$

Step 3: The left side is $\frac{d}{dt}[(4 + t^2)y]$:

$$\frac{d}{dt}[(4 + t^2)y] = 4t$$

Step 4: Integrate:

$$(4+t^2)y = \int 4t\,dt = 2t^2 + C$$

Step 5: Solve for y:

$$y = \frac{2t^2 + C}{4+t^2}$$

□

Example 2.5 (Initial Value Problem)
Find the general solution of the differential equation

$$\frac{dy}{dt} - 2y = 4 - t$$

and solve the initial value problem with $y(0) = 1$. Discuss the behavior as $t \to \infty$.

Solution. This is in standard form with $p(t) = -2$ and $g(t) = 4 - t$.

Step 1: Find the integrating factor:

$$\mu(t) = e^{\int -2dt} = e^{-2t}$$

Step 2-3: Multiply and recognize:

$$\frac{d}{dt}[e^{-2t}y] = e^{-2t}(4-t)$$

Step 4: Integrate the right side using integration by parts:

$$\int e^{-2t}(4-t)dt = \int 4e^{-2t}dt - \int te^{-2t}dt$$

$$= -2e^{-2t} - \left(-\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t}\right)$$

$$= -2e^{-2t} + \frac{1}{2}te^{-2t} + \frac{1}{4}e^{-2t}$$

$$= e^{-2t}\left(-2 + \frac{t}{2} + \frac{1}{4}\right) = e^{-2t}\left(\frac{t}{2} - \frac{7}{4}\right)$$

Therefore:

$$e^{-2t}y = e^{-2t}\left(\frac{t}{2} - \frac{7}{4}\right) + C$$

Step 5: General solution:

$$y = \frac{t}{2} - \frac{7}{4} + Ce^{2t}$$

Initial condition: $y(0) = 1$:

$$1 = \frac{0}{2} - \frac{7}{4} + C \Rightarrow C = 1 + \frac{7}{4} = \frac{11}{4}$$

Particular solution:

$$y = \frac{t}{2} - \frac{7}{4} + \frac{11}{4}e^{2t}$$

Behavior as $t \to \infty$: Since e^{2t} grows exponentially, $y(t) \to +\infty$ as $t \to \infty$. □

Example 2.6 (Homogeneous Linear Equation)
Solve $y' + 3y = 0$ with $y(0) = 2$.

Solution. For a homogeneous equation $y' + p(t)y = 0$, we can solve by separation:

$$\frac{dy}{y} = -p(t)dt$$

Integrating: $\ln|y| = -\int p(t)dt + C$
With $p(t) = 3$:

$$\ln|y| = -3t + C \Rightarrow y = Ae^{-3t}$$

Using $y(0) = 2$: $A = 2$
Therefore: $y = 2e^{-3t}$ □

Remark 2.7 (Structure of Solutions)**.** *The general solution of a first-order linear equation $y' + p(t)y = g(t)$ has the form:*

$$y = y_h + y_p$$

where:

- $y_h = Ce^{-\int p(t)dt}$ *is the general solution to the homogeneous equation*
- y_p *is any particular solution to the nonhomogeneous equation*

As $t \to \infty$:

- *If $p(t) > 0$, then $y_h \to 0$ (transient behavior)*
- *If $p(t) < 0$, then y_h grows exponentially*
- *The long-term behavior is determined by y_p when $p(t) > 0$*

2 Separable Equations

3 Modeling with First Order Differential Equations

4 Differences Between Linear and Nonlinear Equations

Theorem 2.8 (Existence and Uniqueness Theorem for First-Order Linear Equations)
If the functions p and g are continous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation

$$y' + p(t)y = g(t)$$

for each t in I, and that also satisifies the initial condition

$$y(t_0) = y_0$$

where y_0 is an arbitrary prescribed inital value.

Theorem 2.9 (Existence and Uniqueness Theorem for First-Order Linear Equations)
Let the functions f and $\frac{\partial f}{\partial y}$ be continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0.$$

Example 2.10
Find an interval in which the initial value problem

$$ty' + 2y = 4t^2, \quad y(1) = 2$$

has a unique solution. Then do the same when the initial value is changed to $y(-1) = 2$.

Example 2.11
Find an interval in which the initial value problem

$$\frac{dy}{dt} = \frac{3x^2 + 4x + 2}{2(y - 1)}, \quad y(0) = -1$$

has a unique solution. Then do the same when the initial value is changed to $y(0) = 1$.

Example 2.12
Consider the initial value problem

$$y' = y^{\frac{1}{3}}, \quad y(0) = 0$$

5 Autonomous Equations and Population Dynamics

6 Exact Equation and Integrating Factors

Definition 2.13 (Exact Differential Equation)

A first-order differential equation of the form

$$M(x, y) + N(x, y)y' = 0$$

or equivalently written as

$$M(x, y)dx + N(x, y)dy = 0$$

is called an **exact differential equation** if there exists a function $\psi(x, y)$ such that"

$$\frac{\partial \psi}{\partial x} = M(x, y) \tag{1}$$

$$\frac{\partial \psi}{\partial y} = N(x, y) \tag{2}$$

Theorem 2.14 (Test for Exactness)

Let the functions M, N, M_y, and N_x, where subscripts denote partial derivatives, be continuous in the rectangular region $R : \alpha < x < \beta, \gamma < y < \delta$. Then equation

$$M(x, y) + N(x, y)y' = 0$$

is an exact differential equation in R if and only if

$$M_y(x, y) = N_x(x, y)$$

at each point of R. That is, there exists a function ψ satisfying equations

$$\psi_x(x, y) = M(x, y), \quad \psi_y(x, y) = N(x, y),$$

3 Second Order Linear Differential Equations

1 Homogeneous Equations with Constant Coefficients

Definition 3.1

A **second-order linear homogeneous differential equation with constant coefficients** has the general form:

$$ay'' + by' + cy = 0$$

where a, b, and c are real constants with $a \neq 0$.

The key insight for solving these equations is that exponential functions have the property that their derivatives are proportional to themselves, making them natural candidates for solutions.

Theorem 3.2 (Characteristic Equation Method)

To solve the differential equation $ay'' + by' + cy = 0$:

1. **Assume an exponential solution:** Let $y = e^{rt}$ where r is a constant to be determined.
2. **Compute derivatives:**
$$y' = re^{rt}$$
$$y'' = r^2 e^{rt}$$
3. **Substitute into the differential equation:**
$$a(r^2 e^{rt}) + b(re^{rt}) + c(e^{rt}) = 0$$
4. **Factor out e^{rt} (which is never zero):**
$$e^{rt}(ar^2 + br + c) = 0$$
5. **Obtain the characteristic equation:**
$$ar^2 + br + c = 0$$
6. **Solve using the quadratic formula:**
$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The nature of the roots determines the form of the general solution. The **discriminant** $\Delta = b^2 - 4ac$ plays a crucial role in determining the behavior of solutions.

Theorem 3.3 (General Solutions Based on Root Types)
Let r_1 and r_2 be the roots of the characteristic equation $ar^2 + br + c = 0$. Then:

Case 1: Real and Distinct Roots ($\Delta > 0$)

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

where c_1 and c_2 are arbitrary constants determined by initial conditions.

Case 2: Complex Conjugate Roots ($\Delta < 0$) If $r_{1,2} = \alpha \pm \beta i$ where $\alpha = -\frac{b}{2a}$ and $\beta = \frac{\sqrt{4ac-b^2}}{2a}$, then:

$$y(t) = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

This represents oscillatory motion with exponential growth/decay.

Case 3: Repeated Real Root ($\Delta = 0$) If $r_1 = r_2 = r = -\frac{b}{2a}$, then:

$$y(t) = (c_1 + c_2 t)e^{rt}$$

The factor t appears due to the need for linear independence.

Remark 3.4. *The physical interpretation varies by case:*

- **Distinct real roots:** *Exponential growth/decay (no oscillation)*
- **Complex roots:** *Damped or growing oscillations*
- **Repeated roots:** *Critical damping (fastest approach to equilibrium without oscillation)*

Example 3.5

Consider the differential equation $y'' - 3y' + 2y = 0$.

Step 1: Form the characteristic equation:

$$r^2 - 3r + 2 = 0$$

Step 2: Factor or use quadratic formula:

$$(r-1)(r-2) = 0$$

So $r_1 = 1$ and $r_2 = 2$.

Step 3: Since we have real and distinct roots, the general solution is:

$$y(t) = c_1 e^t + c_2 e^{2t}$$

Verification: We can verify this solution by substitution:

$$\begin{aligned} y &= c_1 e^t + c_2 e^{2t} \\ y' &= c_1 e^t + 2c_2 e^{2t} \\ y'' &= c_1 e^t + 4c_2 e^{2t} \end{aligned}$$

Substituting: $y'' - 3y' + 2y = (c_1 e^t + 4c_2 e^{2t}) - 3(c_1 e^t + 2c_2 e^{2t}) + 2(c_1 e^t + c_2 e^{2t}) = 0$

2 Solutions of Linear Homogeneous Equations; the Wronskian

Definition 3.6 (Wronskian)

The Wronskian of two functions f and g is defined as

$$W(f, g)(t) = f(t)g'(t) - g(t)f'(t)$$

Example 3.7

Let $f(t) = e^t$ and $g(t) = e^{2t}$. Then we have:

$$W(f, g)(t) = e^t(2e^{2t}) - e^{2t}(e^t) = 2e^{3t} - e^{3t} = e^{3t}$$

Since $W(f, g)(t) \neq 0$ for all t, the functions f and g are linearly independent.

Theorem 3.8

If f and g are differentiable functions, on an open interval I, and if $W(f, g)(t_0) \neq 0$, for some point $t_0 \in I$, then f and g are linearly independent on I. Moreover, if f and g are linearly dependent on I, then the Wronskian $W(f, g)(t) = 0$ for every $t \in I$.

Theorem 3.9 (Abel's Theorem)
If y_1 and y_2 are two solutions of the second-order linear homogeneous differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

on an interval I, then the Wronskian $W(y_1, y_2)(t)$ satisfies

$$W(y_1, y_2)(t) = W(y_1, y_2)(t_0) \exp\left[-\int_{t_0}^{t} p(s)ds\right]$$

for any $t_0 \in I$.

Theorem 3.10
Let y_1 and y_2 be the solutions of the second-order linear homogeneous differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

where p and q are continuous functions on an interval I. Then the Wronskian $W(y_1, y_2)(t)$ satisfies

$$W(y_1, y_2)(t) = W(y_1, y_2)(t_0) \exp\left[-\int_{t_0}^{t} p(s)ds\right]$$

for any $t_0 \in I$.

3 Complex Roots of the Characteristic Equation

We continue our discussion of the second-order linear differential equation

$$ay'' + by' + cy = 0$$

where a, b, and c are constants. Here we the characteristic equation is

$$ar^2 + br + c = 0$$

and the general solution is

$$y(t) = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

where $\alpha = -\frac{b}{2a}$ and $\beta = \frac{\sqrt{4ac-b^2}}{2a}$. However, $b^2 - 4ac$ could be negative, leading to complex roots. Assuming that they are complex, we can write the roots as

$$r_1 = \lambda + i\mu, \quad r_2 = \lambda - i\mu$$

where λ and μ are real. Therefore the expressions for y are

$$y(t) = e^{\lambda t}(c_1 \cos(\mu t) + c_2 \sin(\mu t))$$

Example 3.11
Find the general solution of the differential equation

$$y'' + y' + 9.25y = 0.$$

Also find the solution that satisfies the initial conditions $y(0) = 2$ and $y'(0) = 0.8$.

Solution. □

Example 3.12
Find the solution of the initial value problem

$$16y'' - 8y' + 145y = 0, \quad y(0) = -2, \quad y'(0) = 1.$$

Solution. □

Example 3.13
Find the general solution of the differential equation

$$y'' + 9y = 0.$$

4 Repeated Roots; Reduction of Order

We know how to solve the equation

$$ay'' + by' + cy = 0 \tag{1}$$

when the roots of the characteristic equation

$$ar^2 + br + c = 0 \tag{2}$$

either are real and different or are complex conjugates. Now we consider the third possibility, namely, that the two roots r_1 and r_2 are equal. This case is transitional between the other two and occurs when the discriminant $b^2 - 4ac$ is zero. Then it follows from the quadratic formula that

$$r_1 = r_2 = -\frac{b}{2a}. \tag{3}$$

The difficulty is immediately apparent; both roots yield the same solution

$$y_1(t) = e^{-bt/(2a)} \tag{4}$$

of the differential equation (1), and it is not obvious how to find a second solution.

For the polynomial $ay'' + by' + cy = 0$ the characteristic equation is

$$ar^2 + br + c = 0$$

If the roots are repeated, i.e., $r_1 = r_2 = r$, then the general solution is

$$y(t) = (c_1 + c_2 t)e^{rt}$$

Example 3.14

Solve the differential equation

$$y'' + 4y' + 4y = 0$$

Example 3.15

Find the solution of the initial value problem

$$y'' - y' + \frac{1}{4}y = 0, \quad y(0) = 2, \quad y'(0) = \frac{1}{3}.$$

5 Nonhomogeneous Equations; Method of Undetermined Coefficients

Now, we could have a situation where the differential equation might also be non-homogeneous which is

$$L[y] = y'' + p(t)y' + q(t)y = g(t)$$

where $p, q.$ and g are given (continous) functions on the open interval I. The equation

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

is very useful in solving this problem.

Theorem 3.16

If Y_1 and Y_2 are two solutions of the non-homogeneous linear differential equation $L[y] = y'' + p(t)y' + q(t)y = g(t)$, then their difference $Y_1 - Y_2$ is a solution of the corresponding homogeneous differential equation (2). If, in addition, y_1 and y_2 form a fundamental set of solutions of equation $L[y] = y'' + p(t)y' + q(t)y = 0$, then

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t)$$

where c_1 and c_2 are certain constants.

Theorem 3.17

The general solution fo the non-homogeneous equation $L[y] = y'' + p(t)y' + q(t)y = g(t)$ can be expressed as

$$y = \phi(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

where y_1 and y_2 form a fundamental set of solutions of the corresponding homogeneous equation $L[y] = y'' + p(t)y' + q(t)y = 0$ and where Y is any particular solution of the non-homogeneous equation.

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