Math 54: Topology

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Introduction

Professor Vladimir Chernov (Tchernov) is the course instructor for this quarter. Office hours, class materials, lecture notes will be available on Canvas. There will be weekly homework which is worth 20% of the final grade, a midterm (40%), and a final exam (40%).

For this course, we will use *Topology* by James R. Munkres (2nd edition). The book is available for purchase online or at the Dartmouth bookstore. You can also access it here.

We will cover the first four chapters of the book, which are as follows:

- **Weeks 1, 2:** Chapter 1 Set Theory and Logic
- **Weeks 3, 4, 5:** Chapter 2 Topological Spaces and Continuous Functions
- **Weeks 6, 7:** Chapter 3 Connectedness and Compactness
- **Weeks 8, 9:** Chapter 4 Countability and Separation Axioms

Contents

1 Set Theory and Logic

1 Fundamental Concepts

We started the class with discussing some basic notation of set theory. For example, $\in, \subset, \cup, \cap, \emptyset$. Here, are usecases of that. For example,

- $a \in A$ means that a is an element of A.
- $A \subset B$ implies that set A is a subset of set B.
- $B = \{x \mid x \text{ is an even integer}\}$ is notation for the set all even integers
- $A \cap B = \{x \mid x \in A \text{ or } x \in B\}$

Example 1.1
If $x^2 < 0 \implies x = 23$. The contrapositive of that would be $x \neq 23 \leftarrow x^2 \geq 0$. The statement and the contrapositive both are true.

Theorem 1.2
Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof. We will prove by showing that $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$ and $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$. Let's start by showing $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$. Suppose, we have $x \in A \cap (B \cup C)$. That means that $x \in A$ and $x \in (B \cup C)$. So that means that $x \in B$ or $x \in C$. Combining them, we get $x \in A$

Now we will prove the other way. Let's start by considering both cases possibe.

- Case α: $x \in A \cap B$
- Case β: $x \in A \cap C$

$\square$

Definition 1.3 (Power of Set)
The set of all subsets of a set A is called the **power set** of A and is denoted by $\mathcal{P}(A)$.

Definition 1.4 (Binary Operation)
A **binary operation** from a set A is function f mapping $A \times A$ into A.

2 Functions

3 Relations

Definition 3.1 (Relation)
A **relation** on a set A is a subset of the cartesian product $A \times A$.

We denote xCy to say that $(x, y) \in C$, and we read this as x is in the relation C to y.

Example 3.2
P is the set of all people $D \subset P \times P$ is given by the equation $D = \{(x, y) \mid x \text{ is a descendant of } y\}$.

Definition 3.3 (Equivalence Relation)
A relation C on a set A is an **equivalence relation** if it is

- Reflexive: $x \sim x, \forall x \in A$
- Symmetric: If $x \sim y$, then $y \sim x$
- Transitive: If $x \sim y$ and $y \sim z$, then $x \sim z$

Example 3.4
Being blood relative is an equivalence relation if you think that every person is a relative of themselves. Being descendant is not an equivalence relatoin though.

Fact 3.5
For equivalence relation C, we generally write $x \sim y$ instead of xCy. Given an equivalence relation $\sim$, an equivalence class is determined by x is denoted by $[x]$ where, $[x] = \{y \in A \mid y \sim x\}$.

Lemma 3.6
Two equivalent classes are either disjoint or equal.

Proof. Let $[x]$ and $[\tilde{x}]$ are two equivalence classes. Suppose we have $y \in [x]$, and $y \in [\tilde{x}]$. Therefore $y \sim x$, and $y \sim \tilde{x}$. Using symmetry, we can write $x \sim y$. Now, we have $x \sim y$, and $y \sim \tilde{x}$. Using transitivity, we can write $x \sim \tilde{x}$. Therefore, we can write $[x] \sim [\tilde{x}]$. Therefore, if we have $[x] \cap [\tilde{x}] \neq \emptyset$, $[x] = [\tilde{x}]$. $\square$

Definition 3.7 (Partition)
A **partition** of a set A is a collection of disjoint non-empty subsets of A whose union is A.

Definition 3.8 (Order Relation)
An **order relation** is a relation $<$ on a set A such that

- Comparability: For every $x, y \in A$ with $x \neq y$, either $x < y$ or $y < x$.
- Nonreflexivity: For no $x \in A$ does the relation $x < x$ hold.
- Transitivity: If $x < y$ and $y < z$, then $x < z$.

As the tilde, $\sim$, for equivalence relations, we generally write $x < y$ instead of $x < y$ for order relations.

Definition 3.9 (Open Interval, Immediate Predecessor and Successor)
If X is a set and $<$ is an order relation on X, and if $a < b$, then b is called an **immediate successor** of a if there does not exist $c \in X$ such that $a < c < b$. Similarly, a is called an **immediate predecessor** of b if there does not exist $c \in X$ such that $a < c < b$. The **open interval** with endpoints a and b is the set $(a, b) = \{x \in X \mid a < x < b\}$.

4 The Integers and the Real Numbers

Definition 4.1 (Binary Relation)
A **binary relation** on a set A is a subset of the cartesian product $A \times A$.

Definition 4.2 (Function)
Function f from a set A to a set B is a relation from A to B such that for each $a \in A$, there is a unique $b \in B$ such that $(a, b) \in f$. We write $f : A \to B$. If $(a, b) \in f$, we write $f(a) = b$.

We assume that we have two binary operations $+$ and $\cdot$ on both A and B, and we have an order relation $<$ on both A and B. Then the following properties hold:

Lemma 4.3
Let $f : A \to B$. If there exist functions $g : B \to A$ and $h : B \to A$ such that $g \circ f = a \forall a$ and $f \circ h = a \forall a$, then f is bijective and $g = h = f^{-1}$.